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QUASI-MONOCHROMATIC WEAKLY NON-LINEAR WAVES IN A LOW-DISPERSION BUBBLE MEDIUM†

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Tyumen

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The propagation of quasi-monochromatic wave packets in a rarefied polydispersed mixture of a weakly compressible liquid with a finite number of fractions of differently sized gas bubbles is considered. Two equations for the modulation waves are derived by the multi-scale method in the cubic approximation in the wave amplitude: the non-linear Schrödinger equation ignoring dissipation effects and the Landau–Ginzburg equation for low dissipation due to the viscosity of the liquid and heat losses associated with bubble vibration. The coefficients of the non-linear Schrödinger equation are investigated to analyse the non-linear (modulational) stability of waves in a monodispersed non-dissipative bubble medium.

A LINEAR dispersion relationship has been previously obtained for acoustic waves in a polydispersed bubble medium without dissipation [1] and for waves in a dissipative medium [2]. The general scheme for deriving the amplitude equations by the asymptotic multiscale method has been described in several monographs (see, e.g. [3]). Modulation equations have been obtained [4] for waves in a monodispersed bubble chamber by Whitham's averaged Lagrangian method [5].

1. THE EQUATIONS OF MOTION IN A NON-DISSIPATIVE MEDIUM

The plane one-dimensional motion of an ideal weakly compressible liquid with a low volume content of spherical gas bubbles ($\alpha_g \ll 1$) under conditions when thermal dissipation and capillary effects can be ignored is described by Iordanskii's equations [1, 6, 7]

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$$d_t\rho + \rho\partial_x v = 0, \rho d_t v + \partial_x p = 0, d_t n_j + n_j\partial_x v = 0$$
$$\rho_l\left[a_j d_t^2 a_j + \frac{3}{2}(d_t a_j)^2\right] = p_j - p, \quad \rho_l - \rho_{l0} = C_l^{-2}(p - p_0), \quad \rho = \rho_l(1-\alpha_g)$$
$$p_j = p_0(a_{j0}/a_j)^{3\varkappa}, \quad \alpha_j = \frac{4}{3}\pi n_j a_j^3, \quad \alpha_g = \sum_{j=2}^{N}\alpha_j, \quad j = 2,\ldots,N \tag{1.1}$$

Here ρ, p and v are the density, the pressure and the velocity of the mixture, a_j, p_j, α_j and n_j are the radius, the pressure, the volume fraction and the number of bubbles of fraction j per unit volume of the mixture ($j = 2, \ldots, N$), ρ_l and C_l are the true liquid density and the velocity of sound in the pure liquid and $\varkappa$ is the polytropic exponent ($\varkappa = \gamma_g$ for adiabatic and $\varkappa = 1$ for isothermal vibrations of the bubbles where γ_g is the gas adiabatic exponent). The zero subscript denotes the unperturbed state, $d_t = \partial_t + v\partial_x$.

An analysis of wave propagation in bubble systems [7] shows that the characteristic scales of variation of the density and the velocity of the medium are respectively given by $\rho_* = \rho_0\alpha_{g0}$ and $v_* = (\alpha_{g0}p_0/\rho_0)^{1/2}$, and the characteristics wave velocity is $C_* = [p_0/(\alpha_{g0}\rho_0)]^{1/2}$. Introducing the dimensionless variables

$$a_j = a_j'/a_{\gamma 0} - 1, \; p = p'/p_0 - 1, \; \rho = (\rho' - \rho_0)/\rho_*, \; v = v'/v_*, \; \xi_j = $$
$$= a_{j0}/a_* \tag{1.2}$$
$$t = t'/t_*, \; x = x'/x_*, \; t_* = a_*(\rho_0/p_0)^{1/2}, \; x_* = a_*\alpha_{g0}^{-1/2},$$
$$b^2 = p_0/(\rho_{l0}C_l^2\alpha_{g0})$$

where the prime denotes dimensional variables and a_* is some representative bubble radius (for a monodispersed medium, this may be a_0), and ignoring quantities of the order of α_{g0} and $|\rho_l - \rho_{l0}|/\rho_{l0}$ compared to 1, we eliminate v from system (1.1) and write the dimensionless equations

$$\partial_x^2 p - \partial_t^2\rho = 0, \; \rho - 1 - b^2 p + \langle(1 + a_j)^3\rangle = 0$$
$$\xi_j^2[(1 + a_j)\partial_t^2 a_j + {}^3/_2(\partial_t a_j)^2] - (1 + a_j)^{-3\varkappa} + p + 1 = 0 \tag{1.3}$$
$$j = 2, \ldots, N$$

Here and henceforth, angle brackets denote the action of a linear operator

$$\langle f_j\rangle = \sum_{j=2}^{N} \nu_j f_j, \quad \nu_j = \frac{\alpha_{j0}}{\alpha_{g0}}, \quad j = 2, \ldots, N \tag{1.4}$$

(ν_j is the proportion of jth fraction bubbles in the total bubble volume).

2. DERIVATION OF THE AMPLITUDE EQUATIONS

Consider Eqs (1.3) and (1.4) in the neighbourhood of the equilibrium state $p = 0$, $\rho = 0$, $a_j = 0$, representing the solutions by asymptotic series in the small parameter ε (the relative perturbation amplitude)

$$p = \sum_{m=1}^{\infty}\varepsilon^m p_m, \quad \rho = \sum_{m=1}^{\infty}\varepsilon^m\rho_m, \quad a_j = \sum_{m=1}^{\infty}\varepsilon^m a_{jm} \quad (j = 2, \ldots, N) \tag{2.1}$$

The dependence of the unknowns on x and t, according to the general idea of the construction of uniformly valid expansions by the multiscale method [3, 5, 8], is viewed as the dependences on the sets of "slow" coordinates and times $\{x_s\}$, $\{t_s\}$, $x_s = \varepsilon^s x$, $t_s = \varepsilon^s t$, $s = 0, 1, 2, \ldots$ All x_s and t_s are assumed to be formally independent, and the differentiation operators are represented as power series in ε

$$\partial_x = \sum_{s=0}^{\infty}\varepsilon^s\partial_{x_s}, \quad \partial_t = \sum_{s=0}^{\infty}\varepsilon^s\partial_{t_s}, \quad \partial_{x_s} = \frac{\partial}{\partial x_s}, \quad \partial_{t_s} = \frac{\partial}{\partial t_s} \tag{2.2}$$

Substituting (2.1) and (2.2) into (1.3) and collecting terms with equal powers of ε, we obtain in the mth approximation the linear non-homogeneous system

$$\begin{gathered}\partial_{x_0}{}^2 p_m - \partial_{t_0}{}^2 \rho_m = f_m,\ \rho_m - b^2 p_m + 3\langle a_{jm}\rangle = g_m \\ \xi_j{}^2 \partial_{t_0}{}^2 a_{jm} + 3\varkappa a_{jm} + p_m = h_{jm},\ j = 2, \ldots, N,\ m = 1, 2, \ldots\end{gathered} \tag{2.3}$$

Here f_m, g_m and h_{jm} are determined from lower approximations.

Let us consider partial solutions of system (2.3) in the form

$$\chi_m = \chi^0_{m0} + \sum_{n>0} (\chi^0_{mn} e^{in\theta} + \text{c. c.}),\quad \theta = kx_0 - \omega t_0,\quad \chi = p, \rho, a_j, f, g, h_j \tag{2.4}$$

where θ, k and ω are the phase, the wave number, and the angular frequency, $\chi_{mn}{}^0$ are the complex amplitudes that depend only on x_s, t_s, $s \geqslant 1$ and the symbol c.c. stands for the term which is the complex conjugate of the first term in parentheses. By the orthogonality of the trigonometric system, we obtain for the nth harmonic in the mth approximation

$$\begin{gathered}n^2 p^0_{mn} (k^2 - 3\omega^2 \langle \Lambda_{jn}^{-1}\rangle - \omega^2 b^2) = n^2\omega^2 (g^0_{mn} - 3\langle \Lambda_{jn}^{-1} h_{jmn}\rangle) - f^0_{mn} \\ a^0_{jmn} = \Lambda_{jn}^{-1} (h^0_{jmn} - p^0_{mn}),\ \rho^0_{mn} = g^0_{mn} - 3\langle \Lambda_{jn}^{-1} h^0_{jmn}\rangle + \\ + [3\langle \Lambda_{jn}^{-1}\rangle + b^2]\, p^0_{mn} \\ \Lambda_{jn} = 3\varkappa - \xi_j{}^2\omega^2 n^2,\ j = 2, \ldots, N,\ m = 1, 2, \ldots, n = 0, 1, \ldots\end{gathered} \tag{2.5}$$

Now suppose that we have the dispersion relationship [1] (see Sec. 3)

$$k^2 = \omega^2 (b^2 + 3\langle \Lambda_{j1}^{-1}\rangle) \tag{2.6}$$

Then for solutions of the form (2.4) to exist it is necessary that the resonance harmonics f, g and h_j satisfy the following conditions:

$$\text{for}\ \ n = 0{:}\ f^0_{m0} = 0,\ m = 1, 2, \ldots \tag{2.7}$$

The zeroth-harmonic solution is obtained up to an arbitrary value of $p_{m0}{}^0$:

$$\begin{gathered}\rho^0_{m0} = g^0_{m0} - \varkappa^{-1} [\langle h^0_{jm0}\rangle - (1 + \varkappa b^2)\, p^0_{m0}], \\ a^0_{jm0} = {}^1/_3 \varkappa^{-1} (h^0_{jm0} - p^0_{m0})\end{gathered} \tag{2.8}$$

$$\text{for}\ \ n = 1{:}\ \omega^2 (g^0_{m1} - 3\langle \Lambda_{j1}^{-1} h_{jm1}\rangle) - f^0_{m1} = 0,\ m = 1, 2, \ldots \tag{2.9}$$

The first-harmonic solution is not unique either. Using relationships (2.5) and (2.9), we obtain

$$\rho^0_{m1} = \omega^{-2} (f^0_{m1} + k^2 p^0_{m1}),\quad a^0_{jm1} = \Lambda_{j1}^{-1} (h^0_{jm1} - p^0_{m1}) \tag{2.10}$$

For all other n, we assume that if (2.6) holds, then $\langle \Lambda_{jn}^{-1}\rangle \neq \langle \Lambda_{j1}^{-1}\rangle$. In this case, a unique solution exists [see (2.5)]:

$$p^0_{mn} = \frac{n^2\omega^2 (g^0_{mn} - 3\langle \Lambda_{jn}^{-1} h^0_{jmn}\rangle) - f^0_{mn}}{3n^2\omega^2 \langle \Lambda_{j1}^{-1} - \Lambda_{jn}^{-1}\rangle},\quad n \geqslant 2 \tag{2.11}$$

Thus, relationships (2.5), (2.8), (2.10) and (2.11) determine the nth harmonics in the mth approximation. The existence conditions (2.7) and (2.9) "control" the behaviour of the complex amplitudes, subjecting them to corresponding differential equations in the "slow" coordinates.

3. THE FIRST APPROXIMATION

Suppose that the main harmonic is the monochromatic signal $p_1 = (p_{11}{}^0 \exp(i\theta) + \text{c.c.})$. In the first approximation, system (2.3) is homogeneous ($f_1 = g_1 = h_{j1} = 0$) and the existence conditions (2.7)

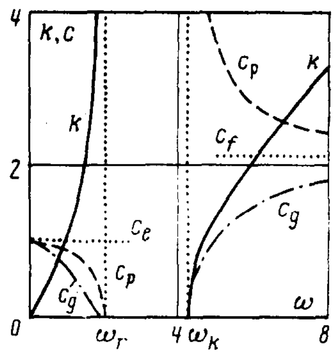

FIG. 1.

and (2.9) are satisfied. The non-trivial solution ($p_{11}^0 \neq 0$) exists only if the dispersion relationship (2.6) holds, and from (2.10) we have

$$\rho_{11}^0 = c_p^{-2} p_{11}^0, \; a_{j11}^0 = -\Lambda_{j1}^{-1} p_{11}^0, \; j = 2, \ldots, N \; (c_p = \omega/k) \tag{3.1}$$

Figures 1 and 2 plot the wave number k (the solid curves), the phase velocity $c_p = \omega/k$ (the dashed curves) and the group velocity $c_g = d\omega/dk$ (the dash–dot curves) of sound as a function of the frequency ω for a monodispersed mixture ($\xi_j \equiv 1$) and a two-fraction mixture ($N = 3$, $\xi_2 = 1$, $\xi_3 = 2$, $\nu_2 = \nu_3 = 0.5$) of water with sufficiently large air bubbles for a pressure $p_0 = 10^5$ Pa, temperature $T_0 = 293$ K ($\rho_{l0} = 1000$ kg/m^3 and $C_l = 1500$ m/s), and volume content $\alpha_{g0} = 2 \times 10^{-4}$ ($\varkappa = \gamma_g = 1.4$ and $b^2 = 0.222$). As we see from (2.6), the equilibrium velocity c_e and the frozen velocity c_f of sound in the mixture are given by

$$c_e = \lim_{\omega \to 0} c_p = \lim_{\omega \to 0} c_g = (b^2 + \varkappa^{-1})^{-1/2}, \quad c_f = \lim_{\omega \to \infty} c_p = \lim_{\omega \to \infty} c_g = b^{-1} \tag{3.2}$$

The vertical dotted lines in the figures mark the boundaries of the ω-regions where $k^2 < 0$ (the so-called "acoustic opacity bands" of the medium [7]). Note that for an M-fraction mixture the number of such bands is M, the lower bound ω_{rj} of each band is equal to the resonance frequency of the jth fraction bubbles $\omega_{rj}^2 = 3\varkappa/\xi_j^2$, and the upper bound ω_{kj} is the root of the equation $k(\omega) = 0$.

4. THE SECOND APPROXIMATION

Substituting relationships (2.1) and (2.2) into (1.3) and collecting terms in ε^2, we obtain from (2.3)

$$f_2 = 2\,(\partial_{t0}\partial_{t1}\rho_1 - \partial_{x0}\partial_{x1}p_1), \quad g_2 = -3\,\langle a_{j1}^2 \rangle \tag{4.1}$$

$$h_{j2} = -\xi_j^2\,[a_{j1}\partial_{t0}^2 a_{j1} + {}^3/_2\,(\partial_{t0}a_{j1})^2 + 2\partial_{t0}\partial_{t1}a_{j1}] + {}^3/_2\varkappa\,(3\varkappa + 1)\,a_{j1}^2$$

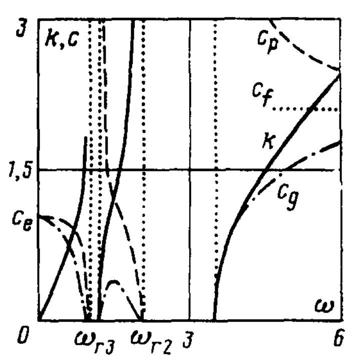

FIG. 2.

From (4.1) and (3.1) we obtain all the harmonics (2.4) of the functions f_2, g_2 and h_{j2}:

$$f_{20}{}^0 = 0,\ f_{21}{}^0 = -2ik\ (c_p{}^{-1}\partial_{t1}p_{11}{}^0 + \partial_{x1}p_{11}{}^0),\ f_{22}{}^0 = 0$$

$$g_{20}{}^0 = -6\ \langle \Lambda_{j1}{}^{-2}\rangle \,|\, p_{11}{}^0\,|^2,\ g_{21}{}^0 = 0,\ g_{22}{}^0 = -3\ \langle \Lambda_{j1}{}^{-2}\rangle\ (p_{11}{}^0)^2$$

$$h^0_{j20} = \Psi_j^{(1)}\Lambda_{j1}^{-2}\,|\,p_{11}{}^\circ\,|^2,\quad \overset{\circ}{h}_{j21} = -2i\omega\xi_j{}^2\Lambda_{j1}^{-1}\partial_{t1}p_{11}{}^\circ,\quad \overset{\circ}{h}_{j22} = \frac{1}{2}\Psi_j^{(2)}\Lambda_{j1}^{-2}(p_{11}{}^\circ)^2$$

$$\Psi_j{}^{(1)} = 3\varkappa(3\varkappa + 1) - \xi_j{}^2\omega^2,\quad \Psi_j{}^{(2)} = 3\varkappa(3\varkappa + 1) + 5\xi_j{}^2\omega^2 \tag{4.2}$$

Condition (2.7) is satisfied, and therefore from (2.8) and (4.2)

$$\rho_{20}{}^0 = -\varkappa^{-1}\ \langle\Psi_j{}^{(3)}\Lambda_{j1}{}^{-2}\rangle\ |\,p_{11}{}^0\,|^2 + c_e{}^{-2}p_{20}{}^0,\ \Psi_j{}^{(3)} = 9\varkappa\ (\varkappa + 1) - \xi_j{}^2\omega^2$$

$$a^0_{j20} = {}^1/_3\varkappa^{-1}\ [\Psi_j{}^{(1)}\Lambda_{j1}^{-2}\ |\ p_{11}{}^0\,|^2 - p_{20}{}^0] \tag{4.3}$$

Condition (2.9), using (2.6), gives the equation

$$\partial_{t1}p_{11}{}^0 + c_g\partial_{x1}p_{11}{}^0 = 0,\quad c_g = d\omega/dk = (c_p{}^{-1} + 3c_p\omega^2\ \langle\xi_j{}^2\Lambda_{j1}{}^{-2}\rangle)^{-1} \tag{4.4}$$

which is consistent with the general theory [3, 5]. Equation (4.4) shows that it is helpful to change to a comoving system of coordinates that moves with the group velocity:

$$p_{11}{}^0\ (x_1,\ t_1,\ x_2,\ t_2,\ .\ .\ .) = p_{11}{}^0\ (\eta_1,\ x_2,\ t_2,\ .\ .\ .),\ \eta_1 = x_1 - c_g t_1 \tag{4.5}$$

Using (4.5), we obtain from (2.10)

$$\rho_{21}{}^0 = -2i\ (\omega c_p)^{-1}\ (1 - c_g c_p{}^{-1})\ \partial_{\eta1}p_{11}{}^0 + c_p{}^{-2}p_{21}{}^0,\ a_{j21}{}^0 =$$
$$= 2ic_g\Lambda_{j1}{}^{-2}\omega\xi_j{}^2\partial_{\eta1}p_{11}{}^0 - \Lambda_{j1}{}^{-1}p_{21}{}^0 \tag{4.6}$$

The harmonic $n = 2$ is not a resonance harmonic, and therefore from (2.11), (2.5), (2.6) and (4.2) we obtain

$$p_{22}{}^0 = -\frac{3}{2}\langle\Psi_j^{(4)}\Lambda_{j1}^{-2}\Lambda_{j2}^{-1}\rangle\ \langle\Lambda_{j1}^{-1} - \Lambda_{j1}^{-1}\rangle^{-1}\,(p_{11}{}^0)^2,\quad \Psi_j^{(4)} = 3\varkappa\,(\varkappa + 1) - \xi_j{}^2\omega^2$$

$$p^0_{22} = c_p^{-2}p^0_{22},\ a^0_{j22} = \Lambda_{j2}^{-1}\,[{}^1/_2\Psi_j^{(2)}\Lambda_{j1}^{-2}\ (p^0_{11})^2 - p^0_{22}] \tag{4.7}$$

5. THE SCHRÖDINGER EQUATION

Substituting (2.1) and (2.2) into (1.3), collecting terms in ε^3 and comparing with (2.3), we obtain

$$f_3 = \partial_{t1}{}^2\rho_1 + 2\partial_{t0}\partial_{t2}\rho_1 + 2\partial_{t0}\partial_{t1}\rho_2 - \partial_{x1}{}^2p_1 - 2\partial_{x0}\partial_{x2}p_1 - 2\partial_{x0}\partial_{x1}p_2 \tag{5.1}$$
$$g_3 = -\langle 6a_{j1}a_{j2} + a_{j1}{}^3\rangle,\ h_{j3} = -\xi_j{}^2\ [a_{j2}\partial_{t0}{}^2a_{j1} + a_{j1}\partial_{t0}{}^2a_{j2} +$$
$$+ 2a_{j1}\partial_{t0}\partial_{t1}a_{j1} + 2\partial_{t0}\partial_{t1}a_{j2} + \partial_{t1}{}^2a_{j1} + 2\partial_{t0}\partial_{t2}a_{j1} +$$
$$+ 3\ (\partial_{t1}a_{j1} + \partial_{t0}a_{j2})\ \partial_{t0}a_{j1}] + 3\varkappa\ (3\varkappa + 1)\ a_{j1}a_{j2} - {}^1/_2\varkappa\ (3\varkappa +$$
$$+ 1)\ (3\varkappa + 2)\ a_{j1}{}^3$$

We thus see that the third-approximation spectrum consists of the harmonics $n = 0, 1, 2, 3$, and $f_{30}{}^0 = 0$. Therefore, the first existence condition (2.7) is satisfied. To satisfy the second existence condition (2.9), we isolate the first harmonic in (5.1). Denoting complex conjugation by a bar, we find from (5.1), using (3.1), (4.5) and (4.6)

$$f_{31}{}^0 = -2ik\ (c_p{}^{-1}\partial_{t2} + \partial_{x2})\ p_{11}{}^0 - (3c_p{}^{-2}c_g{}^2 - 4c_p{}^{-1}c_g + 1)\partial_{\eta1}{}^2p_{11}{}^0 +$$
$$+ 2ik\ (c_p{}^{-1}c_g - 1)\ \partial_{\eta1}p_{21}{}^0$$

$$g_{31}{}^0 = -3\ \langle 2\overline{a^0_{j11}}a^0_{j22} + 2a^0_{j11}a^0_{j20} + a^0_{j11}\ |\ a^0_{j11}\ |\ ^2\rangle,$$

$$h^0_{j31} = -\xi_j{}^2\ [\omega^2\ (\overline{a^0_{j11}}a^0_{j22} - a^0_{j11}a^0_{j20} + 2i\omega\Lambda_{j1}^{-1}\partial_{t2}p_{11}{}^0 -$$

$$- c_g^2)\,(\Lambda_{j1}^{-1} + 4\omega^2\xi_j^2\Lambda_{j1}^{-2})\,\partial_{\eta 1}{}^2 p_{11}{}^0 - 2i\omega c_g \Lambda_{j1}^{-1}\partial_{\eta 1} p_{21}{}^0] +$$
$$+ 3\varkappa\,(3\varkappa + 1)\,[\overline{a_{j11}^0} a_{j22}^0 + a_{j11}^0 a_{j20}^0 - {}^1/_2(3\varkappa + 2)\, a_{j11}^0 \,|\, a_{j11}^0\,|^2] \qquad (5.2)$$

Note that (5.2) contains two undefined quantities $p_{21}{}^0$ and $p_{20}{}^0$ [through $a_{j20}{}^0$, see (4.3)]. The first is automatically eliminated when (5.2) is substituted into (2.9); the second must be determined from supplementary considerations. To this end, we turn to the fourth approximation. We can directly show that

$$f_{40}{}^0 = \partial_{t1}{}^2\rho_{20}{}^0 - \partial_{x1}{}^2 p_{20}{}^0 = \partial_{\eta 1}{}^2\,(c_g{}^2\rho_{20}{}^0 - p_{20}{}^0)$$

Since the fourth-approximation solutions exist only when conditions (2.7) are satisfied ($f_{40}{}^0 = 0$), we may assume without loss of generality that $p_{20}{}^0 = c_g{}^2\rho_{20}{}^0$ (indeed, $p_{20}{}^0$ is determined by quadratic non-linearity). This and (4.3) give

$$p_{20}{}^0 = -\varkappa^{-1}\,(c_g{}^{-2} - c_e{}^{-2})^{-1}\,\langle \Psi_j{}^{(3)}\Lambda_{j1}{}^{-2}\rangle\,|\,p_{11}{}^0\,|^2 \quad (c_e{}^{-2} = b^2 + \varkappa^{-1}) \qquad (5.3)$$

Now, with all the quantities expressed in terms of the complex pressure amplitude $p_{11}{}^0$, we can substitute (5.2) into (2.9) and use (2.6), (3.1), (4.2)–(4.4), (4.7) and (5.3) to obtain

$$i\,(\partial_{t2} + c_g\partial_{x2})\,p_{11}{}^0 + \beta\partial_{\eta 1}{}^2 p_{11}{}^0 + \gamma p_{11}{}^0\,|\,p_{11}{}^0\,|^2 = 0 \qquad (5.4)$$
$$\beta = {}^1/_2 dc_g/dk = {}^1/_2 k^{-1}c_g\,\{1 - c_g{}^2\,[5\,(c_p c_g)^{-1} - 4c_p{}^{-2} + 12\omega^4\,\langle \xi_j{}^4\Lambda_{j1}{}^{-3}\rangle]\}$$
$$\gamma = {}^1/_2 c_g c_p \omega\,\{\varkappa^{-1}\,[\langle \Psi_j{}^{(1)}\Psi_j{}^{(3)}\Lambda_{j1}{}^{-4}\rangle + \varkappa^{-1}\,(c_g{}^{-2} - c_e{}^{-2})^{-1}\,\langle \Psi_j{}^{(3)}\Lambda_{j1}{}^{-}\rangle^2] -$$
$$- {}^3/_2\,[\langle \Psi_j{}^{(5)}\Lambda_{j1}{}^{-4}\rangle - 3\,\langle \Psi_j{}^{(2)}\Psi_j{}^{(4)}\Lambda_{j1}{}^{-4}\Lambda_{j2}{}^{-1}\rangle +$$
$$+ 9\,\langle \Psi_j{}^{(4)}\Lambda_{j1}{}^{-2}\Lambda_{j2}{}^{-1}\rangle^2\,\langle \Lambda_{j2}{}^{-1} - \Lambda_{j1}{}^{-1}\rangle^{-1}]\}$$
$$(\Psi_j{}^{(5)} = 27\varkappa^2\,(\varkappa + 1) + 2\omega^2\xi_j{}^2)$$

Changing to a coordinate system that moves with the group velocity $\eta_2 = x_3 - c_g t_2$ and taking $p_{11}{}^0 = A\,(\eta_1, \eta_2, t_2, \ldots)$, we obtain the non-linear Schrödinger equation (NSE) in standard form [3, 5]

$$i\partial_{t2}A + \beta\partial_{\eta 1}{}^2 A + \gamma A\,|\,A\,|^2 = 0 \qquad (5.5)$$

Figures 3 and 4 plot the frequency dependence of the NSE coefficients γ (the solid curves) and β (the dashed curves) for monodispersed (Fig. 3) and polydispersed (two-fraction, $N = 3$, Fig. 4) mixtures with the parameters $\varkappa = 1.4$ and $b^2 = 0.222$. In Fig. 3, the vertical dash–dot line delimits the acoustic opacity bands of the medium (see Fig. 1), the vertical dotted line corresponding to

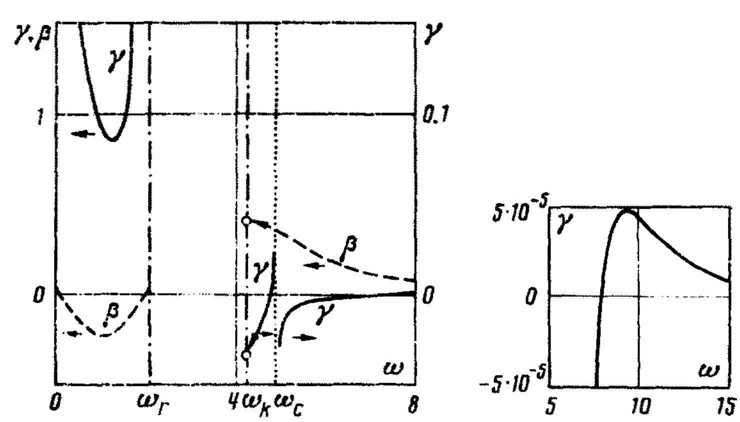

FIG. 3.

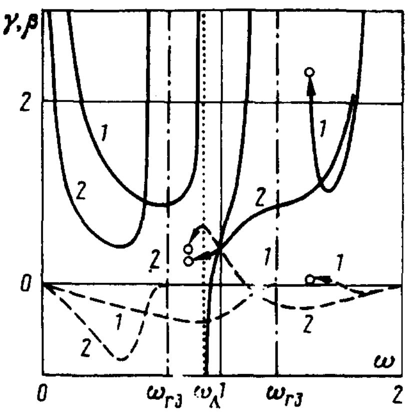

FIG. 4.

$\omega = \omega_c$ is the asymptote of γ [the frequency ω_c is the root of the equation $c_g(\omega) = c_e$ and for $\omega \to \omega_c$ from (5.4) we obtain $|\gamma| \to \infty$; geometrically, ω_c can be obtained from Fig. 1 by continuing the line $c = c_e$ to its intersection with $c = c_g$; at the points $\omega = 0$ and $\omega = \omega_c$, the tangents to $k(\omega)$ have equal slopes].

Curves 1 and 2 in Fig. 3 correspond to $\xi_3 = 1.6$ and $\xi_3 = 3$ with $\xi_2 = 1$, $\nu_2 = \nu_3 = 0.5$. They have been constructed in the frequency range $\omega < \omega_{r2} = (3\varkappa)^{1/2}$. In the first case, we have a singularity in γ for $\omega = \omega_\Lambda$ (the vertical dotted line) when $\langle \Lambda_{j1}^{-1}(\omega_\Lambda) \rangle = \langle \Lambda_{j2}^{1}(\omega_\Lambda) \rangle$ and $|\gamma| \to \infty$ for $\omega \to \omega_\Lambda$ (5.4); in the second case, no such singularity is observed (it exists for the two-fraction mixture with $\xi_2 = 1$, $\nu_2 = \nu_3 = 0.5$ only when $0.5 < \xi_3 < 2$; geometrically (see Fig. 2), the existence of this singularity is equivalent to the existence of the line $k = \lambda\omega$ that crosses two branches of the dispersion relationship $k = k(\omega)$ at the points $\omega = \omega_\Lambda$ and $\omega = 2\omega_\Lambda$, respectively). The vertical dash–dot lines in the figure represent the resonance frequencies of the large-bubble fraction ($\omega_{r3}^2 = 3\varkappa/\xi_3^2$) that correspond to the lower bounds of the low-frequency acoustic opacity bands; the upper bounds of these bands can be determined from the (finite) limiting values of β and γ (the arrows on the curves). Note that the qualitative behaviour of β and γ for polydispersed mixtures in the region $\omega > \omega_{r2}$ is the same as for the monodispersed mixture (Fig. 3).

6. MODULATIONAL INSTABILITY

The question of the non-linear modulational instability of wave packets in a non-dissipative medium can be investigated by analysing the signs of the NSE coefficients [3, 5]. Thus, the case $\beta\gamma < 0$ corresponds to (neutral) stability and the case $\beta\gamma > 0$ corresponds to Benjamin–Feir instability [3], when the NSE has solutions in the form of wave envelope solitons.

Of the greatest interest apparently are the stability regions in parameter space. However, already for a monodispersed mixture, this space is three-dimensional $(\omega, b, \varkappa)$ and its dimension increases by two with the addition of each new bubble fraction (ν_j and ξ_j). The number of zeros and singularities of the NSE coefficients, reflecting the interaction of different branches of the dispersion relationship, also substantially increases in this case (see Figs 3, 4).

As an example, consider the important theoretical case of a monodispersed mixture. The polytropic exponent $\varkappa$ is fixed, and ω and b are arbitrary [the parameter b characterizes the ratio of the carrying phase compressibility to the compressibility of the phase associated with the bubbles; thus for an incompressible carrying phase ($C_1 = 0$) $b = 0$, for a medium without bubbles ($\alpha_{g0} = 0$) $b = \infty$, see (1.2)].

The results of analysis and calculations of the stability regions (ω, b) and (k, b) using relationship (5.4) for $\varkappa = 1.4$ are presented in Figs 5 and 6. The stability region $\beta\gamma < 0$ is not hatched, and the acoustic opacity band ($k^2 < 0$) is shown by diagonal hatching in Fig. 5. At the fundamental signal frequencies $\omega < \omega_r$ the wave envelope soliton is not created ($\beta\gamma < 0$), and therefore if we consider the evolution of perturbations with a given wave number, then from the two branches of the dispersion equation (Fig. 1) $\omega_-(k)$, $\omega_+(k)$ the lower branch ($\omega_-(k) < \omega_r$) is stable for any k and b. The curve $\omega = \omega_k[\omega_k^2 = 3(b^{-2} + \varkappa)]$ (the dash–dot curve in Fig. 5)

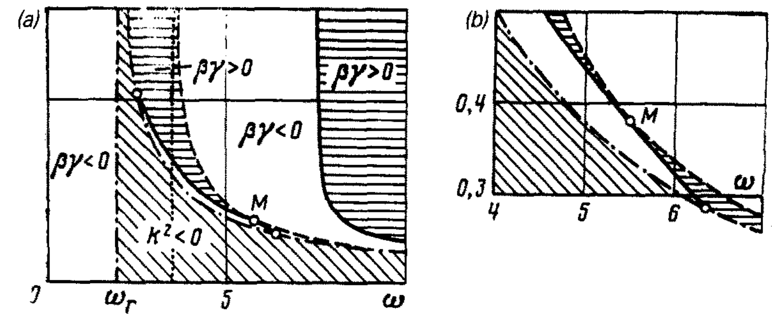

FIG. 5.

corresponds to the line $k = 0$ in Fig. 6 (Fig. 6 plots k corresponding to the upper branch $\omega_+(k) \geqslant \omega_k$ of the dispersion relationship). The dashed curve (in Fig. 5 with the vertical asymptote shown by the dotted line) is the curve $\omega = \omega_c\,[c_g(\omega_c) = c_e$, Fig. 3] with the parametric equation

$$\omega^2 = 3\varkappa z^{-1}(1+2z),\quad k^2 = 3z(1+2z)^2(1+z)^{-2}(1-z)^{-1}$$
$$b^2 = \varkappa^{-1}(1+z+z^2)(1+z)^{-2}(1-z)^{-1},\quad 0 < z < 1$$

The singularity of the coefficient γ on this curve is avoidable only at the point $M[z_M = (3\varkappa+1)^{-1}]$, where the dashed curve touches the solid curve that corresponds to $\gamma = 0$. The second solid curve delimiting the high-frequency instability region corresponds to the second zero of the coefficient γ (see also Fig. 3). The curve in Fig. 6 has a limiting point $k = \sqrt{2}$. Note that the instability generated in the neighbourhood of the curve $\omega = \omega_c$ physically corresponds to the Benney long-wave/short-wave resonance [3].

A general result for mono- and polydispersed mixtures can be obtained by examining the principal terms of the low- and high-frequency asymptotic forms of the coefficients of the NSE (5.4):

$$\beta\,|_{\omega\to 0} \sim -{}^1\!/_2\varkappa^{-2}c_e^4\omega^{-1}\langle\xi_j^2\rangle,\quad \gamma\,|_{\omega\to 0} \sim {}^1\!/_4\varkappa^{-2}(\varkappa+1)^2c_e^2\omega^{-1}\langle\xi_j^2\rangle^{-1} \tag{6.1}$$
$$\beta\,|_{\omega\to\infty} \sim {}^3\!/_2c_f^4\omega^{-3}\langle\xi_j^{-2}\rangle,\quad \gamma\,|_{\omega\to\infty} \sim {}^1\!/_2\varkappa^{-1}c_f^2\omega^{-3}(\langle\xi_j^{-4}\rangle - \langle\xi_j^{-2}\rangle^2)\quad (c_f = b^{-1})$$

For a monodispersed mixture ($\xi_j \equiv 1$), the asymptotic form of γ as $\omega \to \infty$ has the form $\gamma \sim {}^9\!/_4 c_f^2\omega^{-5}$. By Hölder's inequality $\langle\xi_j^{-4}\rangle \geqslant \langle\xi_j^{-2}\rangle^2$, and therefore for mono- and polydispersed mixtures we obtain from (6.1) $\beta\gamma < 0$ for $\omega \to 0$ and $\beta\gamma > 0$ for $\omega \to \infty$. This corresponds to stability of low-frequency waves and instability of high-frequency waves.

7. LOW DISSIPATION. THE LANDAU–GINZBURG EQUATION

Let us now return to the original system of equations (1.1). For a viscous liquid with interacting phases and thermal dissipation, which may be obtained by solving the internal problem of heat conduction in a gas [7], the Rayleigh–Lamb equation and the polytropic oscillation equation are transformed [7] into the following equations:

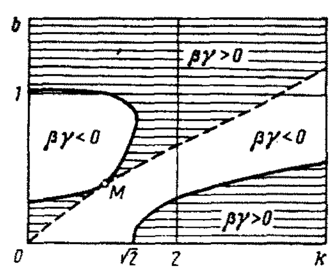

FIG. 6.

$$\rho_l\,[a_j d_t^2 a_j + {}^3/_2\,(d_t a_j)^2] + 4\mu_l a_j^{-1} d_t a_j = p_j - p \tag{7.1}$$

$$p_j^{-1} d_t p_j + 3\gamma_g a_j^{-1} d_t a_j = -3\,(\gamma_g - 1)\, a_j^{-1} p_j^{-1} q_j,\ q_j = -\lambda_g\,(\partial_r T_j)_{r=a_j} \tag{7.2}$$

Here q_j is the heat flux across the phase boundary, T_j is the temperature field inside a jth fraction bubble (r is the radial coordinate reckoned from the centre of the bubble) and μ_l, λ_g are the dynamic coefficient of viscosity of the liquid and the thermal conductivity of the gas.

Let us consider a typical case when the characteristic oscillation frequencies ω_* are such that the characteristic thickness of the unsteady thermal boundary layer in the bubbles $\delta_T = (\nu_g^{(T)}/\omega_*)^{1/2}$ ($\nu_g^{(T)}$ is the thermal diffusivity) is much less than the size of the bubble ($\delta_T \ll a_{j0}$) and the oscillations of the bubbles are nearly adiabatic ($\varkappa = \gamma_g$). Let us estimate the orders of the terms in Eq. (7.2). Both terms on the left-hand side are of the same order ($\varepsilon\omega_*$), while the term on the right-hand side is $\sim \varepsilon a_{j0}^{-1} p_0^{-1} \lambda_g T_0 \delta_T^{-1} (q_j \sim \varepsilon \lambda_g \delta_T^{-1} T_0)$. The ratio of the latter to the former is of the order of $\delta_T/a_{j0} \ll 1$ ($\nu_g^{(T)} = \lambda_g/(\rho_{g0} c_{pg}$, $p_0 = \rho_{g0} R_g T_0$, $R_g/c_{pg} = 1 - \gamma_g^{-1}$; ρ_g, R_g, c_{pg} are the density, the gas constant, and the heat capacity at constant pressure). Thus, if $\delta_T/a_{j0} \sim \varepsilon^m \ll 1$, then up to the $(m+1)$th approximation in ε we can use the value of q_j obtained by solving the linearized heat conduction problem and linearize the right-hand side of Eq. (7.2):

$$\partial_t T_j = \nu_g^{(T)} r^{-2} \partial_r\,(r^2 \partial_r T_j) + (\rho_{g0} c_{pg})^{-1} d_t p_j,\quad T_j\,|_{r=a_{j0}} = T_0 \tag{7.3}$$

(for the gas inside the bubble we use the homobaric condition $\partial_r p_j = 0$ [7]). The solution of problem (7.3) is well known and in the case $\delta_T/a_{j0} \ll 1$ it has the form

$$q_j = \left(\frac{\nu_g^{(T)}}{\pi}\right)^{1/2} \int_{-\infty}^{t} (t-\tau)^{-1/2}\, d_\tau p_j(\tau)\, d\tau \tag{7.4}$$

In view of the above $p_j/p_0 = (a_j/a_{j0})^{-3\gamma_g}[1 + O(\delta_T/a_{j0})]$. Thus, making the appropriate substitution in (7.4) and integrating Eq. (7.2) with the linearized right-hand side, we obtain

$$\frac{p_j}{p_0} = \left(\frac{a_j}{a_{j0}}\right)^{-3\gamma_g} \left[1 + \frac{9\gamma_g(\gamma_g - 1)}{a_{j0}} \left(\frac{\nu_g^{(T)}}{\pi}\right)^{1/2} \int_{-\infty}^{t} (t-\tau)^{-1/2} \left[\frac{a_j}{a_{j0}} - 1\right] d\tau\right] \tag{7.5}$$

Using the dimensionless variables (1.2), we can finally write the Rayleigh–Lamb equation, which takes into account, in the linear approximation, viscous and thermal dissipation, in the form [see (1.3)]

$$\xi_j^2\,[(1 + a_j)\,\partial_t^2 a_j + {}^3/_2\,(\partial_t a_j)^2] - (1 + a_j)^{-3\gamma_g} + p + 1 =$$

$$= -\alpha_\mu \partial_t a_j + \frac{\alpha_T}{\xi_j \pi^{-1/2}} \int_{-\infty}^{t} (t-\tau)^{-1/2}\, a_j(\tau)\, d\tau \tag{7.6}$$

$$\alpha_\mu = 4\mu_l/(p_0 t_*),\ \alpha_T = 9\gamma_g\,(\gamma_g - 1)\,(\nu_g^{(T)} t_*/a_*^2)^{1/2}$$

Let us now consider a procedure for deriving the amplitude equations. To ensure that dissipation appears only in the third approximation, we should formally take $\alpha_\mu = \varepsilon^2 \alpha_\mu^{(0)}$, $\alpha_T = \varepsilon^2 \alpha_T^{(0)}$; $\alpha_\mu^{(0)}$, $\alpha_T^{(0)} = O(1)$. Then only the function h_{j3} changes [see (5.1)], and on the right-hand side of (6.6) we should take $t = t_0$, $a_j = a_{j1}$. Seeing that $a_{j1} = (a_{j11}^0 \exp(i\theta) + \text{c.c.})$, and evaluating the integral (the Fresnel integral), we find an additional term (associated with dissipation) to the first harmonic h_{j31}^0 in (5.2):

$$h_{j31}^{0(d)} = [i\omega\alpha_\mu^{(0)} + (1 + i)\,\xi_j^{-1}\,(2\omega)^{-1/2}\alpha_T^0]\,a_{j11}^0$$

Using (3.1) and the existence condition (2.9), we obtain an additional term in the NSE (5.5). This additional term reduces the NSE to the Landau–Ginzburg equation (LGE) [3]

$$i\partial_{t_2} A + \alpha A + \beta \partial_{\eta_1}^2 A + \gamma A\,|A|^2 = 0,\quad \alpha = \alpha_R + i\alpha_I \tag{7.7}$$

$$\alpha_R = \frac{3}{2\sqrt{2}}\,\omega^{1/2} c_p c_g \alpha_T^{(0)} \langle \xi_j^{-1} \Lambda_{j1}^{-2} \rangle,\quad \alpha_I = \frac{3}{2}\,\omega^2 c_p c_g \alpha_\mu^{(0)} \langle \Lambda_{j1}^{-2} \rangle + \alpha_R$$

At high frequencies we have

$$\alpha_R|_{\omega\to\infty} \sim \frac{3}{2\sqrt{2}} c_f^2 \alpha_T^{(0)} \omega^{-7/2} \langle \xi_j^{-5} \rangle, \quad \alpha_I|_{\omega\to\infty} \sim \frac{3}{2} c_f^2 \alpha_\mu^{(0)} \omega^{-2} \langle \xi_j^{-4} \rangle + \alpha_R$$

Comparison with formulas (6.1) shows that at high frequencies the viscosity of the liquid starts playing a dominant role, suppressing the high-frequency non-linear instability.

In conclusion we note that the LGE theory has been developed much less than the NSE theory, because in general the LGE is a non-integrable equation [3]. Nevertheless, the LGE, like the NSE, arises in the description of many physical systems [3] and is an important object for research, analogies, etc.

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RENORMALIZATION GROUP METHOD FOR THE PROBLEM OF CONVECTIVE DIFFUSION WITH IRREVERSIBLE SORPTION†

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The renormalization group method is used to analyse the propagation of a thin solute slug in a seepage flow with account of diffusion and sorption processes. Sorption is assumed to be partially irreversible and is described by an isotherm with a hysteresis loop. A general technique is developed for analysing the problem. Calculations for the self-similar case are presented and the results are shown to be sufficiently accurate compared with the exact solution.

A NUMBER of problems in the theory of solute transport by seepage flow require consideration of the irreversibility of sorption in the porous medium. Irreversible retention of the solute is particularly

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